SOLUTIONS TO ARTIN'S *ALGEBRA*, 2ND ED.
CH. 1 – MATRICES

COLIN COMMANS

Contents

§1 - The Basic Operations

1.1

What are the entries a_{21} and a_{23} of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 0 & 9 & 4 \end{bmatrix}$?

Solution.

$$a_{21} = 2, \; a_{23} = 8$$

1.2

Determine the products AB and BA for the following values of A and B:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -8 & -4 \\ 9 & 5 \\ -3 & -2 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix}$$

Solution.
(i)

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -8 & -4 \\ 9 & 5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -8+18-9 & -4+10-6 \\ -24+27-3 & -12+15-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -8 & -4 \\ 9 & 5 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -8-12 & -16-12 & -24-4 \\ 9+15 & 18+15 & 27+5 \\ -3-6 & -6-6 & -9-2 \end{bmatrix} = \begin{bmatrix} -20 & -28 & -28 \\ 24 & 33 & 32 \\ -9 & -12 & -11 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6+12 & -4+8 \\ 6+6 & -4+4 \end{bmatrix} = \begin{bmatrix} 18 & 4 \\ 12 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6-4 & 24-8 \\ 3+2 & 12+4 \end{bmatrix} = \begin{bmatrix} 2 & 16 \\ 5 & 16 \end{bmatrix}$$

1.3

Let $A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ be a row vector and let $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector. Compute the products AB and BA.

Solution.

$$AB = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$BA = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & \dots & a_n b_1 \\ \vdots & \ddots & \vdots \\ a_1 b_n & \dots & a_n b_n \end{bmatrix}$$

1.4

Verify the associative law for the matrix product $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

Solution.

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 8 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 38 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 38 \\ 14 \end{bmatrix}$$

1.5

Let A, B, and C be matrices of sizes $\ell \times m$, $m \times n$, and $n \times p$. How many multiplications are required to compute the product AB? In which order should the triple product ABC be computed, so as to minimize the number of multiplications required?

Solution.
If we let $D = AB$, then we have $d_{ij} = a_{i1} b_{1j} + \cdots + a_{im} b_{mj}$. Therefore each entry requires m multiplications, and there are $\ell \times n$ entries of D. Hence we need $m \times (\ell \times n)$ multiplications.

By the reasoning above, $(AB)C$ will first require mln multiplications, followed by nlp giving $mln + nlp$ multiplications in total. Similarly, $A(BC)$ will need $nmp + mlp$ multiplications. Hence $A(BC)$ should be computed first if and only if $\ell n(m + p) \leq mp(n + \ell)$.

1.6

Compute $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}^n$

Solution.

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ & 1 \end{bmatrix}$$

Repeatedly applying this result gives

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & na \\ & 1 \end{bmatrix}$$

(formally prove by induction, if you like)

1.7

Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n$, and prove it by induction.

Solution.
We claim

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ & 1 & n \\ & & 1 \end{bmatrix}$$

Proof.
The base case is $n = 1$, which clearly works after substituting.
Now suppose the claim holds for some n. Then note

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} &\overset{IH}{=} \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ & 1 & n \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+0+0 & 1+n+0 & 1+n+\frac{1}{2}n(n+1) \\ 0+0+0 & 0+1+0 & 0+1+n \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & n+1 & \frac{2(n+1)}{2} + \frac{n(n+1)}{2} \\ & 1 & n+1 \\ & & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & (n+1) & \frac{1}{2}(n+1)(n+2) \\ & 1 & (n+1) \\ & & 1 \end{bmatrix}
\end{aligned}$$

which completes the induction. □

1.8

Compute the following products by block multiplication:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]; \qquad \left[\begin{array}{c|cc} 0 & 1 & 2 \\ \hline 0 & 1 & 0 \\ 3 & 0 & 1 \end{array}\right]\left[\begin{array}{c|cc} 1 & 2 & 3 \\ \hline 4 & 2 & 3 \\ 5 & 0 & 4 \end{array}\right]$$

Solution.
(i) We first denote our 2×2 matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad D' = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

Then we want to compute

$$\begin{bmatrix} A & B \\ I & D \end{bmatrix}\begin{bmatrix} A' & I \\ I & D' \end{bmatrix} = \begin{bmatrix} AA' + BI & AI + BD' \\ IA' + DI & II + DD' \end{bmatrix} = \begin{bmatrix} AA' + B & A + BD' \\ A' + D & I + DD' \end{bmatrix}$$

We have

$$AA' = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad BD' = \begin{bmatrix} 5 & 16 \\ 1 & 3 \end{bmatrix} \quad DD' = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

Hence our product is

$$\begin{bmatrix} 2 & 8 & 6 & 17 \\ 0 & 2 & 1 & 4 \\ 1 & 3 & 2 & 3 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

(ii) Again we denote

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad B' = \begin{bmatrix} 2 & 3 \end{bmatrix} \quad C' = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad D' = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$$

and want to compute

$$\begin{bmatrix} 0 & B \\ C & I \end{bmatrix}\begin{bmatrix} 1 & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + BC' & 0B' + BD' \\ C1 + IC' & CB' + ID' \end{bmatrix} = \begin{bmatrix} BC' & BD' \\ C + C' & CB' + D' \end{bmatrix}$$

We have

$$BC' = 14 \quad BD' = \begin{bmatrix} 2 & 11 \end{bmatrix} \quad CB' = \begin{bmatrix} 0 & 0 \\ 6 & 9 \end{bmatrix}$$

Hence our product is

$$\begin{bmatrix} 14 & 2 & 11 \\ 4 & 2 & 3 \\ 8 & 6 & 13 \end{bmatrix}$$

1.9

Let A, B be square matrices.

(a) When is $(A+B)(A-B) = A^2 - B^2$?

(b) Expand $(A+B)^3$

Solution.
(a) We always have

$$(A+B)(A-B) = (A+B)A + (A+B)(-B) = AA + BA - AB - BB$$

Hence it equals $A^2 - B^2$ if and only if $AB = BA$.

(b)

$$\begin{aligned}(A+B)^3 = [(A+B)(A+B)](A+B) &= [(A+B)A + (A+B)B](A+B) \\ &= [A^2 + BA + AB + B^2](A+B) \\ &= [A^2 + BA + AB + B^2]A + [A^2 + BA + AB + B^2]B \\ &= A^3 + BA^2 + ABA + B^2A + A^2B + BAB + AB^2 + B^3\end{aligned}$$

1.10

Let D be the diagonal matrix with diagonal entries $d_1, \ldots, d_n$, and let $A = (a_{ij})$ be an arbitrary $n \times n$ matrix. Compute the products DA and AD.

Solution.
If we write the rows and columns of A as

$$A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_n \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & \ldots & C_n \end{bmatrix}$$

then we have

$$DA = \begin{bmatrix} d_1R_1 \\ d_2R_2 \\ \cdots \\ d_nR_n \end{bmatrix} \qquad AD = \begin{bmatrix} d_1C_1 & d_2C_2 & \ldots & d_nC_n \end{bmatrix}$$

1.11

Prove that the product of upper triangular matrices is upper triangular.

Solution.

Proof.
Let A and B be two upper triangular matrices, i.e. square matrices of the form

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ & \ddots & \vdots \\ & & b_{nn} \end{bmatrix}$$

Note that $a_{ij} = 0$ if $j > i$, and that this property is equivalent to being upper triangular. Hence for $C = AB$, it suffices to show $i > j \implies c_{ij} = 0$.

Let $i > j$. Then we have

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj} = \sum_{\ell=1}^{i-1} a_{i\ell}b_{\ell j} + \sum_{\ell=i}^{n} a_{i\ell}b_{\ell j}$$

However, note

- when $i > \ell$, we have $a_{i\ell} = 0$ and each term in the first summation is zero.
- when $\ell \geq i > j$, we have $b_{\ell j} = 0$ and each term in the second summation is zero.

Hence

$$\sum_{\ell=1}^{i-1} a_{i\ell}b_{\ell j} = \sum_{\ell=i}^{n} a_{i\ell}b_{\ell j} = 0 \implies c_{ij} = 0$$

□

1.12

In each case, find all 2×2 matrices that commute with the given matrix.

$$\text{(a)} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$$

Solution.

(a)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}; \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

The two matrices are equal iff $b = c = 0$. Therefore we have commuting set

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbb{R} \right\}$$

(b)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}; \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

The matrices are equal iff $c = 0$ and $a = d$. Therefore we have commuting set

$$\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

(c)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 2a & 6b \\ 2c & 6d \end{bmatrix}; \qquad \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 6c & 6d \end{bmatrix}$$

The matrices are equal iff $b = c = 0$. Therefore we have commuting set

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \;\middle|\; a, d \in \mathbb{R} \right\}$$

(d)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 3a+b \\ c & 3c+d \end{bmatrix}; \qquad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix}$$

The matrices are equal iff $c = 0$ and $a = d$. Therefore we have commuting set

$$\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$$

(e)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 2a & 3a+6b \\ 2c & 3c+6d \end{bmatrix}; \qquad \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+3c & 2b+3d \\ 6c & 6d \end{bmatrix}$$

The matrices are equal iff $c = 0$ and $3a + 4b - 3d = 0$. Therefore we have commuting set

$$\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\middle|\; 3a + 4b - 3d = 0 \right\}$$

1.13

A square matrix A is *nilpotent* if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent, then $I + A$ is invertible. Do this by finding the inverse.

Solution.

Proof.
Let A be a nilpotent matrix. We claim

$$(I + A)^{-1} = I - A + A^2 - A^3 + \cdots \pm A^{k-1}$$

To verify this, we compute

$$\begin{aligned}(I + A)(I - A + A^2 - \cdots \pm A^{k-1}) &= (I - A + A^2 - \cdots \pm A^{k-1}) + (A - A^2 + A^3 - \cdots \pm A^k)\\ &= I + (-A + A) + (A^2 - A^2) + \cdots + (\pm A^{k-1} \mp A^{k-1}) \pm A^k\\ &= I + 0 + 0 + \cdots + 0 \pm 0 = I\end{aligned}$$

□

1.14

Find infinitely many matrices B such that $BA = I_2$, when

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

and prove that there is no matrix C such that $AC = I_3$.

Solution.
Choose any real number x. Then

$$\begin{bmatrix} x & -x-1 & 2-x \\ x & 1-x & -x-1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2x + (-x-1) + (2-x) & 3x - 2(x+1) + (2-x) \\ 2x + (1-x) + (-x-1) & 3x + 2(1-x) - (x+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

However note for any 2×3 matrix we have

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix} = \begin{bmatrix} 2a + 3x & 2b + 3y & 2c + 3z \\ a + 2x & b + 2y & c + 2z \\ a + x & b + y & c + z \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the last equality to hold, in particular we need $a + x = 0$, i.e. $a = -x$. But then we also need $0 = a + 2x = (-x) + 2x = x$. Thus $a = x = 0$. But now $2a + 3x = 0 \neq 1$, so the last equality cannot hold.

1.15

With A arbitrary, determine the products $e_{ij}A, Ae_{ij}, e_jAe_k, e_{ii}Ae_{jj}$, and $e_{ij}Ae_{k\ell}$.

Solution.
In all problems, A is $m \times n$.
(i) Let $U = e_{ij}$ and $C = UA$. Then $u_{k\ell} = 1$ if $k\ell = ij$ and zero otherwise. Now

$$c_{k\ell} = u_{k1}a_{1\ell} + \cdots + u_{kj}a_{j\ell} + \cdots + u_{kn}a_{n\ell} = \begin{cases} a_{j\ell} & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

in other words, the ith row of $e_{ij}A$ is the jth row of A, with all other rows being zero.

(ii) Let $U = e_{ij}$ and $C = AU$. Then

$$c_{k\ell} = a_{k1}u_{1\ell} + \cdots + a_{ki}u_{i\ell} + \cdots + a_{km}u_{m\ell} = \begin{cases} a_{ki} & \text{if } \ell = j \\ 0 & \text{if } \ell \neq j \end{cases}$$

in other words, the jth column of Ae_{ij} is the ith column of A, with all other columns being zero.

(iii) Ae_k is the kth column of A, but since e_j is defined to be column vector, $e_j(Ae_k)$ is only valid when A only has one row, in which case (Ae_k) is a scalar and we are just scaling e_j. If we let e_j be a row vector instead, then we are taking the jth entry of Ae_k, which is simply the entry a_{jk}.

(iv) From (i), we have $e_{ii}A$ is the ith row of A and all zeros elsewhere. From (ii), we have $(e_{ii}A)e_{jj}$ is the jth column of that matrix and all zeros elsewhere. This leaves only a_{ij} untouched, therefore $e_{ii}Ae_{jj} = a_{ij}e_{ij}$.

(v) From (i), $e_{ij}A$ takes jth row of A and puts it in the ith row of a zero matrix. From (ii), $(e_{ij}A)e_{k\ell}$ then takes the kth column of that matrix (whose only untouched entry is a_{jk}), and puts that in the ℓth column of a zero matrix. In other words, the entry a_{jk} is now at row i and column ℓ. Hence $e_{ij}Ae_{k\ell} = a_{jk}e_{i\ell}$.

2.1

For the reduction of the matrix M, determine the elementary matrices corresponding to each operation. Compute the product P of these elementary matrices and verify that PM is indeed the end result.

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 6 & 10 \\ 1 & 2 & 5 & 2 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 3 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Solution.
We perform $M' = E_6 E_5 E_4 E_3 E_2 E_1 M$, where

- $E_1 : R_2 \mapsto R_2 - R_1$
- $E_2 : R_3 \mapsto R_3 - R_1$
- $E_3 : R_2 \leftrightarrow R_3$
- $E_4 : R_1 \mapsto R_1 - R_2$
- $E_5 : R_3 \mapsto \frac{1}{5} R_3$
- $E_6 : R_2 \mapsto R_2 - R_3$

which correspond to elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Now we have

$$P = E_6 E_5 E_4 E_3 E_2 E_1 = \begin{bmatrix} 2 & 0 & -1 \\ -0.8 & -0.2 & 1 \\ -0.2 & 0.2 & 0 \end{bmatrix}$$

and indeed

$$PM = \begin{bmatrix} 2 & 0 & -1 \\ -0.8 & -0.2 & 1 \\ -0.2 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 1 & 5 \\ 1 & 1 & 2 & 6 & 10 \\ 1 & 2 & 5 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.2

Find all solutions of the system of equations $AX = B$ when

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \text{(a)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{(b)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{(c)} \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

Solution.
We first row reduce

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 3 & 0 & 0 & 4 \\ 1 & 2 & 1 & 1 \\ 1 & -4 & -2 & 2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 1 & 2 & 1 & 1 \\ 1 & -4 & -2 & 2 \end{bmatrix} \xrightarrow{E_3} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 2 & 1 & -1/3 \\ 1 & -4 & -2 & 2 \end{bmatrix}$$

$$\xrightarrow{E_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4/3 & b_1 \\ 0 & 2 & 1 & -1/3 & b_2 \\ 0 & -4 & -2 & 2/3 & b_3 \end{array}\right] \xrightarrow{E_5} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 1/2 & -1/6 \\ 0 & -4 & -2 & 2/3 \end{bmatrix} \xrightarrow{E_6} \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 1/2 & -1/6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$E_1 : R_1 \leftrightarrow R_2 \qquad E_2 : R_1 \mapsto \tfrac{1}{3}R_1 \qquad E_3 : R_2 \mapsto R_2 - R_1$$

$$E_4 : R_3 \mapsto R_3 - R_1 \qquad E_5 : R_2 \mapsto \tfrac{1}{2}R_2 \qquad E_6 : R_3 \mapsto R_3 - 4R_2$$

We do these operations on each B to get our final system:

(a)

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence we have the system

$$\begin{cases} x_1 + \frac{4}{3}x_4 = 0 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{6}x_4 = 0 \\ 0 = 0 \end{cases}$$

and solution set

$$\left\{ \begin{bmatrix} -\frac{4}{3}x_4 \\ \frac{1}{6}x_4 - \frac{1}{2}x_3 \\ x_3 \\ x_4 \end{bmatrix} \;\middle|\; x_3, x_4 \in \mathbb{R} \right\}$$

(b)

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix}$$

Hence we have the system

$$\begin{cases} x_1 + \frac{4}{3}x_4 = \frac{1}{3} \\ x_2 + \frac{1}{2}x_3 - \frac{1}{6}x_4 = \frac{1}{3} \\ 0 = -1 \end{cases}$$

which has no solutions

(c)

$$B = \begin{bmatrix}0\\2\\2\end{bmatrix} \to \begin{bmatrix}2\\0\\2\end{bmatrix} \to \begin{bmatrix}\frac{2}{3}\\0\\2\end{bmatrix} \to \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\2\end{bmatrix} \to \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{4}{3}\end{bmatrix} \to \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\\frac{4}{3}\end{bmatrix} \to \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\0\end{bmatrix}$$

Hence we have the system

$$\begin{cases} x_1 + \frac{4}{3}x_4 = \frac{2}{3} \\ x_2 + \frac{1}{2}x_3 - \frac{1}{6}x_4 = -\frac{1}{3} \\ 0 = 0 \end{cases}$$

and solution set

$$\left\{ \begin{bmatrix} \frac{2}{3} - \frac{4}{3}x_4 \\ -\frac{1}{3} + \frac{1}{6}x_4 - \frac{1}{2}x_3 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_3, x_4 \in \mathbb{R} \right\}$$

2.3

Find all solutions of the equation $x_1 + x_2 + 2x_3 - x_4 = 3$.

Solution.
We have three degrees of freedom in choosing x_1, x_2, x_3. The equation forces $x_4 = x_1 + x_2 + 2x_3 - 3$. Hence we have solution set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + x_2 + 2x_3 - 3 \end{bmatrix} \middle| \; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

2.4

Determine the elementary matrices used in the row reduction example below, and verify that their product is A^{-1}.

$$\begin{bmatrix}1 & 5\\2 & 6\end{bmatrix} \xrightarrow{E_1} \begin{bmatrix}1 & 5\\0 & -4\end{bmatrix} \xrightarrow{E_2} \begin{bmatrix}1 & 5\\0 & 1\end{bmatrix} \xrightarrow{E_3} \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Solution.
We have

$$E_1 = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix} \qquad E_2 = \begin{bmatrix}1 & 0\\0 & -\frac{1}{4}\end{bmatrix} \qquad E_3 = \begin{bmatrix}1 & -5\\0 & 1\end{bmatrix}$$

and their product is

$$E_3E_2E_1I = \begin{bmatrix}-1.5 & 1.25\\0.5 & -0.25\end{bmatrix} = -\frac{1}{4}\begin{bmatrix}6 & -5\\-2 & 1\end{bmatrix} = A^{-1}$$

2.5

Find inverses of the following matrices:

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Solution.
(i)

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(ii)

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

(iii) Here we use the formula $(ABC)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

2.6

The matrix below is based on the Pascal triangle. Find its inverse.

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

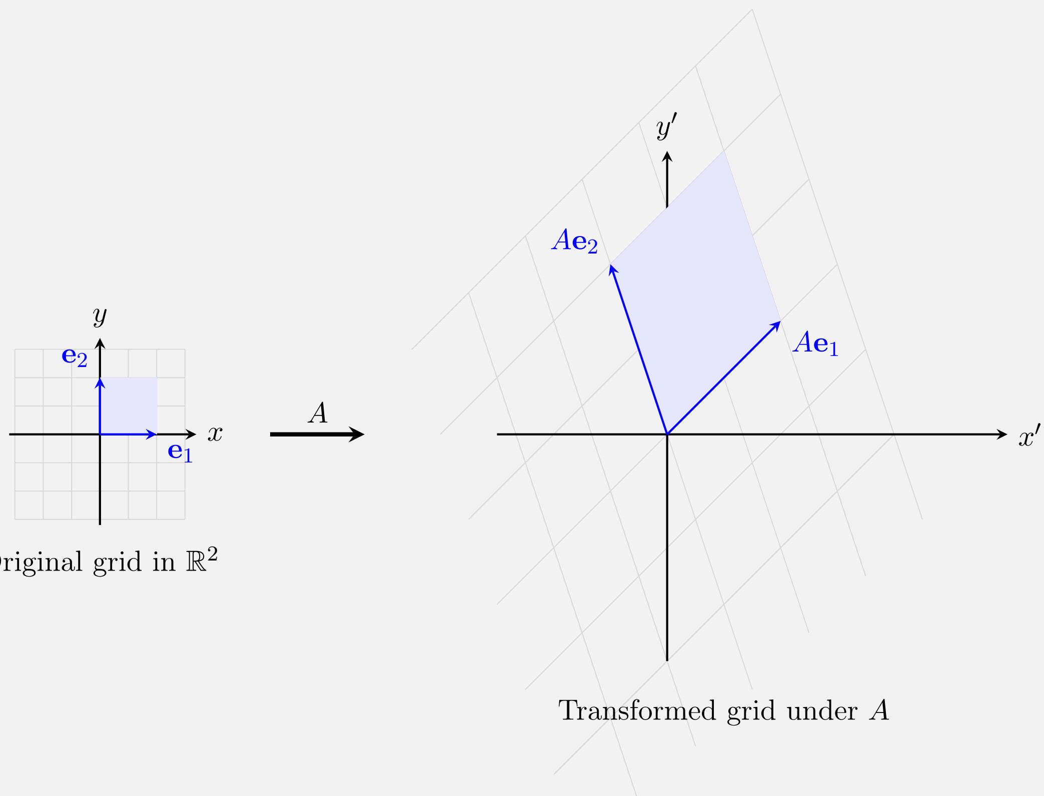
Solution.

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ 1 & -2 & 1 & & \\ -1 & 3 & -3 & 1 & \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

2.7

Make a sketch showing the effect of multiplication by the matrix $A = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$ on the plane $\mathbb{R}^2$.

Solution.



Original grid in $\mathbb{R}^2$

Transformed grid under A

2.8

Prove that if a product AB of $n \times n$ matrices is invertible, so are the factors A and B.

Solution.

Proof.
Suppose that AB has an inverse C. Then we have by associativity

$$I = (AB)C = A(BC) \implies A^{-1} = BC$$

and

$$I = C(AB) = (CA)B \implies B^{-1} = CA$$

Hence both A and B are invertible. □

2.9

Consider an arbitrary system of linear equations $AX = B$, where A and B are real matrices.

(a) Prove that if the system of equations $AX = B$ has more than one solution then it has infinitely many.

(b) Prove that if there is a solution in the complex numbers there there is also a real solution.

Solution.

(a) *Proof.*
Suppose a system $AX = B$ has two distinct solutions, say $\hat{X}$ and $\tilde{X}$. Then for any real number α between 0 and 1, we have by linearity

$$\begin{aligned} A(\alpha\hat{X} + (1-\alpha)\tilde{X}) = A(\alpha\hat{X}) + A((1-\alpha)\tilde{X}) &= \alpha A\hat{X} + (1-\alpha)A\tilde{X} \\ &= \alpha B + (1-\alpha)B \\ &= (\alpha + 1 - \alpha)B = B \end{aligned}$$

And since there are infinitely many values $\alpha \in [0,1]$, we have infinitely many solutions. □

[Of course, we could allow α to be any real number, but the geometric interpretation of picking a point on the line between $\hat{X}$ and $\tilde{X}$ is visualized (and motivated) easier.]

(b) *Proof.*
Let $\hat{X}$ be a complex solution to $AX = B$. We can decompose $\hat{X}$ into its real and imaginary parts, say $\hat{X} = Y + iZ$ for real matrices Y and Z. Now note that

$$B = A\hat{X} = A(Y + iZ) = AY + iAZ$$

since A, B, Y, Z are all matrices with real-number entries, We have AY and AZ both real matrices, and therefore

$$\begin{cases} AY = \text{real part}(B) = B \\ AZ = \text{imaginary part}(B) = 0 \end{cases}$$

In particular, we have Y is a real solution to our system. □

2.10

Let A be a square matrix. Show that if the system $AX = B$ has a unique solution for some particular column vector B, then it has a unique solution for all B.

Solution.
Suppose that $AX = \overline{B}$ has a unique solution for a fixed $\overline{B}$, say $X = \hat{X}$. Now by Theorem 1.2.21 it suffices to show that the system $AX = 0$ only has the solution $X = 0$.

Suppose that $AX = 0$. Now note that

$$\overline{B} = \overline{B} + 0 = A\hat{X} + AX = A(\hat{X} + X)$$

But $AY = \overline{B}$ if and only if $Y = \hat{X}$, thus we have

$$\hat{X} + X = \hat{X} \implies X = 0$$

3.1

A matrix B is *symmetric* if $B = B^t$. Prove that for any square matrices B, BB^t, and $B + B^t$ are symmetric, and that if A is invertible, then $(A^{-1})^t = (A^t)^{-1}$.

Solution.
Let B be a square matrix. Then note that

$$(BB^t)^t = (B^t)^t B^t = BB^t$$

$$(B + B^t)^t = B^t + (B^t)^t = B^t + B = B + B^t$$

Hence BB^t and $B + B^t$ are both symmetric.

Now let A be invertible. Then

$$A^t(A^{-1})^t = (A^{-1}A)^t = I^t = I$$

Thus $(A^{-1})^t$ is the inverse of A^t, i.e. $(A^t)^{-1} = (A^{-1})^t$

3.2

Let A and B be symmetric $n \times n$ matrices. Prove that the product AB is symmetric if and only if $AB = BA$.

Solution.
$\Longrightarrow$: Suppose AB is symmetric. Then by definition

$$AB = (AB)^t = B^t A^t = BA$$

$\Longleftarrow$: Suppose $AB = BA$. Then

$$(AB)^t = B^t A^t = BA = AB$$

3.3

Suppose we first make a row operation, and then a column operation, on a matrix A. Explain what happens if we switch the order of these operations, making the column operation first, followed by the row operation.

Solution.
Let E be our row operation and E' be our column operation. Then the result of these operations in the first case is $(EA)E'$. The result of swapping the order gives $E(AE')$. However, by associativity of matrix multiplication we have $(EA)E' = E(AE')$. Hence changing the order of the two operations has no effect.

3.4

How much can a matrix be simplified if both row and column operations are allowed?

Solution.
We claim we can reduce any nonzero matrix to the block form

$$\left[\begin{array}{c|c} I & \\ \hline & \end{array}\right]$$

Proof.
Let A be an arbitrary nonzero matrix. Then we can reduce to row echelon form A' using row operations. Next, we can permute columns, since those are column operations, to get all the pivots to be all together. This gives the form

$$\left[\begin{array}{c|c} I & B \\ \hline & \end{array}\right]$$

Finally we can make column operations to eliminate each entry of B one element at a time using the appropriate column of I. This leaves us with the desired form. □

4.1

Evaluate the following determinants:

$$\text{(a)} \begin{bmatrix} 1 & i \\ 2-i & 3 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 0 & 9 & 7 & 4 \end{bmatrix}$$

Solution.

(a)

$$\det \begin{bmatrix} 1 & i \\ 2-i & 3 \end{bmatrix} = 3 - (2i+1) = 2 - 2i$$

(b)

$$\det \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = -1 - 1 = -2$$

(c)

$$\det \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + 1 \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 4 - 0 - 1 = 3$$

(d)

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 8 & 6 & 3 & 0 \\ 0 & 9 & 7 & 4 \end{bmatrix} = 1 \det \begin{bmatrix} 2 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 7 & 4 \end{bmatrix} - 5 \det \begin{bmatrix} 0 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 7 & 4 \end{bmatrix} + 8 \det \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 9 & 7 & 4 \end{bmatrix} - 0 \det \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & 3 & 0 \end{bmatrix}$$
$$= 24 - 0 + 0 - 0 = 24$$

4.2

Verify the rule $\det AB = (\det A)(\det B)$ for the matrices

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix}$$

Solution.

$$\det A = \det \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = 8 - 3 = 5 \qquad \det B = \det \begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix} = -2 - 5 = -7$$

$$\det AB = \det \begin{bmatrix} 17 & -4 \\ 21 & -7 \end{bmatrix} = -119 + 84 = -35 = 5 \cdot -7$$

4.3

Compute the determinant of the following $n \times n$ matrix using induction on n:

$$\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

Solution.
We claim the determinant is $n+1$.

Proof.
We use strong induction on n. For $n = 1$, we have $\det \begin{bmatrix} 2 \end{bmatrix} = 2$. For $n = 2$, we have $\det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$. Now we expand along the first column (twice) to get

$$\begin{aligned}
\det \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} &= 2 \det \underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{n-1} + \det \begin{bmatrix} -1 & & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \\
&= 2 \det \underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{n-1} + (-1) \det \underbrace{\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & & \\ & & 2 & -1 \\ & & -1 & 2 \end{bmatrix}}_{n-2} \\
&\qquad - (-1) \det \underbrace{\begin{bmatrix} & & & \\ -1 & \ddots & & \\ & & 2 & -1 \\ & & -1 & 2 \end{bmatrix}}_{\text{top row zeros}} \\
&\overset{IH}{=} 2n - (n-1) + 0 \\
&= n + 1
\end{aligned}$$

□

4.4

Let A be an $n \times n$ matrix. Determine $\det(-A)$ in terms of $\det A$.

Solution.
We claim $\det(-A) = (-1)^n \det A$.

Proof.
We induct on n. When $n = 1$, we have $\det \begin{bmatrix} -a \end{bmatrix} = -a = -\det \begin{bmatrix} a \end{bmatrix}$. Assuming the result holds for $(n-1) \times (n-1)$ matrices and letting $B = -A$, we can apply the formula

$$\begin{aligned}
\det(B) &= b_{11} \det(B_{11}) + (-1)b_{21} \det(B_{21}) + \cdots + (-1)^{n-1} b_{n1} \det(B_{n1}) \\
&= -a_{11} \det(-A_{11}) - (-1)a_{21} \det(-A_{21}) - \cdots - (-1)^{n-1} a_{n1} \det(-A_{n1}) \\
&\overset{IH}{=} -1[a_{11}(-1)^{n-1} \det A_{11} + (-1)a_{21}(-1)^{n-1} \det A_{21} + \cdots + (-1)^{n-1} a_{n1}(-1)^{n-1} \det A_{n1}] \\
&= (-1)^n [a_{11} \det A_{11} + (-1)a_{21} \det A_{21} + \cdots + (-1)^{n-1} \det A_{n1}] \\
&= (-1)^n \det A
\end{aligned}$$

which completes the induction. □

4.5

Use row reduction to prove that $\det A^t = \det A$.

Solution.

Proof.
Let A be a square matrix. Consider one step of row reduction, say $EA = A'$. Note that the corresponding column operation of A^t is simply E^t, as

$$A^t E^t = (EA)^t = A'^t$$

Now suppose we have shown that $\det(A') = \det(A'^t)$. Then

$$\det(E) \det(A) = \det(EA) = \det(A') = \det(A'^t) = \det(A^t E^t) = \det(A^t) \det(E^t)$$

Note that if E interchanges rows or multiplies a row by a scalar, then E is symmetric and $E = E^t$ (see Equation (1.2.4)). If E adds a multiple of a row to another, then E^t also does the same operation and by Corollary 1.4.13 we have $\det(E) = \det(E^t)$. Hence we have $\det(A) = \det(A^t)$ after canceling the $\det(E)$ terms.

Now by induction it suffices to prove the case when A is row reduced. Since A is square, either $A = I$ or the bottom row is all zeros. If $A = I$, then clearly $\det(A) = \det(A^t)$. Otherwise, let A have a row of all zeros. Then by Theorem 1.4.10(c) we have $\det(A) = 0$. Hence A is not invertible, so A^t is not invertible either. Thus $\det(A^t) = 0$ as well. □

4.6

Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$, if A and D are square blocks.

Solution.

Proof.
We first rewrite our given matrix

$$\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & D \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

We next compute the determinant of these three matrices:
Claim 1: $\det \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} = \det D$.

Letting I be $n \times n$, we induct on n. Clearly for $n = 0$ we have $\begin{bmatrix} I_0 & 0 \\ 0 & D \end{bmatrix} = D$. Otherwise, note that

$$\begin{bmatrix} I_n & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} 1 & & \\ & I_{n-1} & 0 \\ & 0 & D \end{bmatrix}$$

and by expanding by minors on the first column gives

$$\det \begin{bmatrix} I_n & 0 \\ 0 & D \end{bmatrix} = 1 \cdot \det \begin{bmatrix} I_{n-1} & 0 \\ 0 & D \end{bmatrix} + 0 + \cdots + 0 = \det D$$

where the last equality is our inductive hypothesis.

Claim 2: $\det \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} = 1$.

Note that we can generalize this claim to say the determinant of an upper triangular matrix is the product of the diagonal entries. However in our particular case note

$$\begin{bmatrix} I_n & B \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & & \overline{B} \\ & I_{n-1} & \underline{B} \\ & 0 & I \end{bmatrix}$$

where $\overline{B}$ is the first row of B and $\underline{B}$ is the remaining block of B. Now by the same argument of Claim 1, induction and expanding by minors along the first column, we have our result.

Claim 3: $\det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \det A$.

If A is $k \times k$, then we can make $2k$ row and column swaps to get our matrix in the form $\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$ (send row 1 to row $k+1$, 2 to $k+2$, etc. and the same for columns). Thus by Theorem 1.4.10(b) and Claim 1 we have

$$\det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = (-1)^{2k} \det \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} = \det A$$

Combining these claims gives our result. □

5.1

Write the following permutations as products of disjoint cycles:

$$(12)(13)(14)(15) \quad (123)(234)(345) \quad (1234)(2345) \quad (12)(23)(34)(45)(51)$$

Solution. NB: Each column applies one cycle, working right to left:

(i)

$$\left.\begin{aligned} 1 \mapsto 5 \mapsto 5 \mapsto 5 \mapsto 5 \\ 2 \mapsto 2 \mapsto 2 \mapsto 2 \mapsto 1 \\ 3 \mapsto 3 \mapsto 3 \mapsto 1 \mapsto 2 \\ 4 \mapsto 4 \mapsto 1 \mapsto 3 \mapsto 3 \\ 5 \mapsto 1 \mapsto 4 \mapsto 4 \mapsto 4 \end{aligned}\right\} \implies (15432)$$

(ii)

$$\left.\begin{aligned} 1 \mapsto 1 \mapsto 1 \mapsto 2 \\ 2 \mapsto 2 \mapsto 3 \mapsto 1 \\ 3 \mapsto 4 \mapsto 2 \mapsto 3 \\ 4 \mapsto 5 \mapsto 5 \mapsto 5 \\ 5 \mapsto 3 \mapsto 4 \mapsto 4 \end{aligned}\right\} \implies (12)(45)$$

(iii)

$$\left.\begin{aligned} 1 \mapsto 1 \mapsto 2 \\ 2 \mapsto 3 \mapsto 4 \\ 3 \mapsto 4 \mapsto 1 \\ 4 \mapsto 5 \mapsto 5 \\ 5 \mapsto 2 \mapsto 3 \end{aligned}\right\} \implies (12453)$$

(iv)

$$\left.\begin{aligned} 1 \mapsto 5 \mapsto 4 \mapsto 3 \mapsto 2 \mapsto 1 \\ 2 \mapsto 2 \mapsto 2 \mapsto 2 \mapsto 3 \mapsto 3 \\ 3 \mapsto 3 \mapsto 3 \mapsto 4 \mapsto 4 \mapsto 4 \\ 4 \mapsto 4 \mapsto 5 \mapsto 5 \mapsto 5 \mapsto 5 \\ 5 \mapsto 1 \mapsto 1 \mapsto 1 \mapsto 1 \mapsto 2 \end{aligned}\right\} \implies (2345)$$

1

5.2

Let p be the permutation (1342) of four indices.

(a) Find the associated permutation matrix P.

(b) Write p as a product of transpositions and evaluate the corresponding matrix product.

(c) Determine the sign of p.

Solution.

(a)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

$$p = (1342) = (12)(14)(13)$$

$$\rightsquigarrow \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(12)} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{(14)} \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(13)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{(142)} \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(13)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P}$$

(c) As p can be written as the product of three transpositions, it is odd. Another way to see this is calculating $\det P = -1$.

5.3

Prove that the inverse of a permutation matrix P is its transpose.

Solution.

Proof.
Consider a permutation matrix $P = \sum_i e_{pi,i}$. Then we have (using formula (1.1.23)),

$$P^t P = \left(\sum_j e_{j,pj}\right)\left(\sum_i e_{pi,i}\right) = \sum_{i,j} e_{j,pj} e_{pi,i} = \underbrace{\sum_{i \neq j} e_{j,pj} e_{pi,i}}_{=\,0} + \sum_i e_{i,pi} e_{pi,i} = \sum_i e_{i,i} = I$$

□

5.4

What is the permutation matrix associated to the permutation of n indices defined by $p(i) = n - i + 1$? What is the cycle decomposition of p? What is its sign?

Solution.
We have associated permutation matrix

$$P = \begin{bmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

We can write this in cycles

$$p = (1\ n)(2\ n\text{-}1)\ldots(\lfloor \tfrac{n+1}{2} \rfloor\ \lceil \tfrac{n+1}{2} \rceil)$$

which has sign $(-1)^{n+1}$.

5.5

In the text, in the products qp and pq of the permutations $p = (341)(25)$ and $q = (1452)$ were seen to be different. However, both products turned out to be 3-cycles. Is this an accident?

Solution.
There is nothing special about the product being 3-cycles. Consider two similar-looking permutations

$$p' = (431)(25) \text{ and } q' = (1532)$$

Then we have $p'q' = (12435)$ and $q'p' = (14235)$, which are different but not 3-cycles.

However, it is true in general that if qp is a k-cycle, then pq will also be a k-cycle. Using some group-theoretic notation, we can write

$$pq = pq(pp^{-1}) = p(qp)p^{-1}$$

so if we write $qp = (a_1, \ldots, a_k)$, we then have

$$p(a_1, \ldots, a_k)p^{-1} = (p(a_1), \ldots, p(a_k))$$

6.1

(a) Compute the determinants of the following matrices by expansion on the bottom row:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) Compute the determinants of these matrices using the complete expansion.

(c) Compute the cofactor matrices of these matrices, and verify Theorem 1.6.9 for them.

Solution.

(a)

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -3\det[2] + 4\det[1] = -6 + 4 = -2$$

$$\det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} = 0\det \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} - 2\det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + 1\det \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = 0(-6) - 2(-2) + 1(2) = 6$$

$$\det \begin{bmatrix} 4 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \det \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} + \det \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix} + \det \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} = 1(1) + 1(-9) + 1(5) = -3$$

$$\det \begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} b & c \\ 0 & 1 \end{bmatrix} - \det \begin{bmatrix} a & c \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} = b - (a - c) - b = c - a$$

(b)

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 3 \cdot 2 = -2$$

$$\begin{aligned} \det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix} &= 1 \cdot 4 \cdot 1 + 1 \cdot 2 \cdot 0 + 2 \cdot 2 \cdot 2 - 2 \cdot 4 \cdot 0 - 1 \cdot 2 \cdot 2 - 1 \cdot 2 \cdot 1 \\ &= 4 + 0 + 8 - 0 - 4 - 2 = 6 \end{aligned}$$

$$\begin{aligned} \det \begin{bmatrix} 4 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} &= 4 \cdot 1 \cdot 1 + -1 \cdot -2 \cdot 1 + (1 \cdot 1 \cdot -1) - 1 \cdot 1 \cdot 1 - (4 \cdot -2 \cdot -1) - (-1 \cdot 1 \cdot 1) \\ &= 4 + 2 - 1 - 1 - 8 + 1 = -3 \end{aligned}$$

$$\begin{aligned} \det \begin{bmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} &= a \cdot 0 \cdot 1 + b \cdot 1 \cdot 1 + c \cdot 1 \cdot 1 - c \cdot 0 \cdot 1 - a \cdot 1 \cdot 1 - b \cdot 1 \cdot 1 \\ &= 0 + b + c - 0 - a - b = c - a \end{aligned}$$

(c)

$$\operatorname{cof}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\right) = \begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}^t = \begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$$

$$\operatorname{cof}\left(\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 2\\ 0 & 2 & 1\end{bmatrix}\right) = \begin{bmatrix}0 & -2 & 4\\ 3 & 1 & -2\\ -6 & 2 & 2\end{bmatrix}^t = \begin{bmatrix}0 & 3 & -6\\ -2 & 1 & 2\\ 4 & -2 & 2\end{bmatrix}$$

$$\operatorname{cof}\left(\begin{bmatrix}4 & -1 & 1\\ 1 & 1 & -2\\ 1 & -1 & 1\end{bmatrix}\right) = \begin{bmatrix}-1 & -3 & -2\\ 0 & 3 & 3\\ 1 & 9 & 5\end{bmatrix}^t = \begin{bmatrix}-1 & 0 & 1\\ -3 & 3 & 9\\ -2 & 3 & 5\end{bmatrix}$$

$$\operatorname{cof}\left(\begin{bmatrix}a & b & c\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}\right) = \begin{bmatrix}-1 & 0 & 1\\ c-b & a-c & b-a\\ b & c-a & -b\end{bmatrix}^t = \begin{bmatrix}-1 & c-b & b\\ 0 & a-c & c-a\\ 1 & b-a & -b\end{bmatrix}$$

which after inspecting the inverses, we see the theorem ($A^{-1} = \frac{1}{\det A}\operatorname{cof}(A)$) holds:

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2 & -1\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix} \qquad \begin{bmatrix}1 & 1 & 2\\ 2 & 4 & 2\\ 0 & 2 & 1\end{bmatrix}^{-1} = \begin{bmatrix}0 & \frac{1}{2} & -1\\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{2} & \frac{5}{6}\end{bmatrix}$$

$$\begin{bmatrix}4 & -1 & 1\\ 1 & 1 & -2\\ 1 & -1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{3} & 0 & -\frac{1}{3}\\ 1 & -1 & -3\\ \frac{2}{3} & -1 & -\frac{5}{3}\end{bmatrix} \qquad \begin{bmatrix}a & b & c\\ 1 & 0 & 1\\ 1 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}-\frac{1}{c-a} & \frac{c-b}{c-a} & \frac{b}{c-a}\\ 0 & -1 & 1\\ \frac{1}{c-a} & \frac{b-a}{c-a} & -\frac{b}{c-a}\end{bmatrix}$$

6.2

Let A be an $n \times n$ matrix with integer entries a_{ij}. Prove that A is invertible, and that its inverse A^{-1} has integer entries, if and only if $\det A = \pm 1$.

Solution.

Proof.
$\Longleftarrow$: Suppose that $\det A = \pm 1$.
By the cofactor matrix theorem, we have A is invertible and $A^{-1} = \frac{1}{\alpha} C$, where C is the cofactor matrix of A and $\alpha = \det A = \pm 1$. Note that for each entry of C we have

$$c_{ij} = (-1)^{i+j} \det A_{ji}$$

but A has integer entries and the determinant is a function of adding and multiplying its entries (seen, e.g., via the complete expansion formula (1.6.4)). Thus C has integer entries, and therefore $A^{-1} = \pm C$ also has integer entries.

$\Longrightarrow$: Suppose that $\det A \neq \pm 1$.
If $\det A = 0$, then A is not invertible and we are done. Otherwise, we may assume $\det A$ is not -1, 0, or 1. By the same argument as above, A is invertible and its cofactor matrix C has integer coefficients. However, now $A^{-1} = \frac{1}{\det A} C$. If $\det A$, which is necessarily an integer, does not divide every entry of C, then A^{-1} will have at least one noninteger entry and we are done.

We now assume the remaining possibility, namely that $\alpha := \det A$ divides every entry of C, for the sake of contradiction. If we write $C = \alpha C'$, where C' has integer entries, then note the cofactor matrix theorem also tells us that $CA = \alpha I$. But now

$$\alpha I = CA = (\alpha C')A = \alpha (C'A) \implies \alpha^n = \det(\alpha I) = \det(\alpha C' A) = \alpha^n \det(C'A)$$

Hence after dividing by α^n (which we know is nonzero) from both sides we get $1 = (\det C')(\det A)$. But C' has integer entries, so in particular $\det C'$ is an integer. Therefore we are multiplying two integers to get 1, forcing $\det A = \pm 1$ and we get a contradiction. Thus α does not divide all entries of C and we have A^{-1} with at least one noninteger entry. □

[A nice one-liner of forward direction is that if A and A^{-1} both have integer entries then their determinants must also be integers yet $I = AA^{-1} \implies 1 = \det I = (\det A)(\det A^{-1})$]

M.1

Let a $2n \times 2n$ matrix be given in the form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where each block is an $n \times n$ matrix. Suppose that A is invertible and that $AC = CA$. Use block multiplication to prove that $\det M = \det(AD - CB)$. Give an example to show that this formula need not hold if $AC \neq CA$.

Solution.

Proof.
Note that we can write

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

By the same argument/result in Exercise 4.6, we have

$$\det \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} = 1, \quad \det \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} = (\det A)(\det(D - CA^{-1}B)), \quad \det \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = 1$$

Hence

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = (\det A)(\det(D - CA^{-1}B)) = \det(AD - ACA^{-1}B) = \det(AD - CB)$$

where the final equality comes from $AC = CA$. □

For a counterexample in the more general case, consider

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 2 \\ -2 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Note that A is invertible and

$$AC = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \neq \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} = CA$$

We then get

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -2 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} = 23$$

But

$$\det(AD - CB) = \det\left(\begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} 0 & 6 \\ -4 & -2 \end{bmatrix} \right) = \det \begin{bmatrix} 3 & -6 \\ 5 & -3 \end{bmatrix} = 21$$

M.2

Let A be an $m \times n$ matrix with $m < n$. Prove that A has no left inverse by comparing A to the square $n \times n$ matrix obtained by adding $(n-m)$ rows of zeros at the bottom.

Solution.

Proof.
Suppose otherwise, i.e. there exists B such that $BA = I_n$. Then using block notation consider the $n \times n$ matrix $A' = \begin{bmatrix} A \\ Z \end{bmatrix}$, where Z is the $(n-m) \times n$ matrix of all zeros. Note that in particular A' is not invertible. However by block matrix rules we have

$$\begin{bmatrix} B & Z^t \end{bmatrix} \begin{bmatrix} A \\ Z \end{bmatrix} = BA + Z'Z = I_n$$

Hence A' has an inverse, a contradiction. □

M.3

The *trace* of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}$$

Show that $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, that $\operatorname{tr} AB = \operatorname{tr} BA$, and that if B is invertible, then $\operatorname{tr} A = \operatorname{tr} BAB^{-1}$.

Solution.

Proof.
(i) Let $C = A + B$. Then $c_{ij} = a_{ij} + b_{ij}$, and so

$$\operatorname{tr} C = c_{11} + \cdots + c_{nn} = (a_{11} + b_{11}) + \cdots + (a_{nn} + b_{nn}) = (a_{11} + \cdots + a_{nn}) + (b_{11} + \cdots + b_{nn}) = \operatorname{tr} A + \operatorname{tr} B$$

(ii) Let $C = AB$ and $D = BA$. Then $c_{ij} = a_{i1}b_{1j} + \cdots + a_{in}b_{nj}$ and similarly $d_{ij} = b_{i1}a_{1j} + \cdots + b_{in}a_{nj}$. Then

$$\begin{aligned} \operatorname{tr} C &= c_{11} + c_{22} + \cdots + c_{nn} \\ &= (a_{11}b_{11} \quad + \cdots + \quad a_{1n}b_{n1}) \\ &\quad + (a_{21}b_{12} \quad + \cdots + \quad a_{2n}b_{n2}) \\ &\quad + \ldots \\ &\quad + (a_{n1}b_{1n} \quad + \cdots + \quad a_{nn}b_{nn}) \\ &= (b_{11}a_{11} + \cdots + b_{1n}a_{n1}) + \cdots + (b_{n1}a_{1n} + \cdots + b_{nn}a_{nn}) \\ &= d_{11} + \cdots + d_{nn} = \operatorname{tr} D \end{aligned}$$

where the third equality comes from regrouping the terms vertically.

(iii) Applying (ii), we get

$$\operatorname{tr}[BAB^{-1}] = \operatorname{tr}[B(AB^{-1})] = \operatorname{tr}[(AB^{-1})B] = \operatorname{tr}[A(B^{-1}B)] = \operatorname{tr} A$$

□

M.4

Show that the equation $AB - BA = I$ has no solution in real $n \times n$ matrices A and B.

Solution.

Proof.
Suppose otherwise, i.e. $AB - BA = I$ for some square matrices A and B. Notice that $\operatorname{tr} -C = -\operatorname{tr} C$ follows almost immediately from the definition of trace. Now using this and the properties of M.3, we have

$$n = \operatorname{tr} I = \operatorname{tr}(AB - BA) = \operatorname{tr} AB + \operatorname{tr} -BA = \operatorname{tr} BA - \operatorname{tr} BA = 0$$

which is a contradiction. □

M.5

Write the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices, using as few as you can, and prove that your expression is as short as possible.

Solution.
Consider the row reduction

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{-0.5R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

We can write these row operations as $E_3 E_2 E_1 A = I$, where

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Therefore we have $A = E_1^{-1} E_2^{-1} E_3^{-1}$, with inverses

$$E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Note that these are indeed elementary matrices. We claim three matrices is the fewest needed.

Proof.
Each elementary matrix is a row operation, so it suffices to prove that we only need three row operations to reduce A to I. Note that we need the 2 and 3 entries to both become zero without multiplying the whole row by zero, hence we need at least two row-addition operations. Furthermore, the determinant of A is -2, but the row-addition operations will always have determinant 1. Hence at least one scaling row operation will be necessary. Therefore we need at least three row operations in total, which is a bound we have achieved above. Hence three row operations is indeed the minimum. □

M.6

Determine the smallest integer n such that every invertible 2×2 matrix can be written as a product of at most n elementary matrices.

Solution.
We claim $n = 4$.

Proof.
We start out with a general invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to row-reduce this to the identity. Consider cases:

- If $a = 0$, this forces $bc \neq 0$. Then we have

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \xrightarrow{\frac{1}{c}R_1} \begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix} \xrightarrow{\frac{1}{b}R_2} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{d}{c}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- If $a \neq 0$, then we can do

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\frac{1}{a}R_1} \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \xrightarrow{R_2 - cR_1} \begin{bmatrix} 1 & b/a \\ 0 & d - bc/a \end{bmatrix} \xrightarrow{\frac{a}{ad-bc}R_2} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - \frac{b}{a}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As this works in general, we have an upper bound of four row operations required for any invertible matrix. Now consider the specific matrix

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$$

Clearly we can row reduce this in four row operations by following the $a = 0$ case above. We claim we cannot do any better. First, note that the left side needs to look like $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which can be achieved with a row-swap and row-scaling, or by two row-additions. In either case, we need two row operations to get the left side properly reduced. Furthermore, neither pair of operations result in a reduced right side, so we need to perform a row-addition and row-scaling to fully reduce to the identity. This gives a lower bound of four row operations, which is achieved in the above process. Therefore $n = 4$. □

M.7

(a) Prove the *Vandermonde determinant*:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$$

(b) Prove that an analogous formula for $n \times n$ matrices, using appropriate row operations to clear out the first column.

(c) Use the Vandermonde determinant to prove that there is a unique polynomial $p(t)$ of degree n that takes arbitrary prescribed values at $n+1$ points $t_0, \dots, t_n$.

Solution.

(a) *Proof.*
Since adding multiples of rows does not change the determinant (Thm 1.4.10(a)), note that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-ab & c^2-ac \end{bmatrix} = \det \begin{bmatrix} b-a & c-a \\ (b-a)b & (c-a)c \end{bmatrix}$$

where we first add $-aR_2$ to R_3 and then $-aR_1$ to R_2. The second equality is just expanding along the first column. Furthermore, we can pull scalars out of whole rows and columns when taking the determinant (Thm 1.4.10(c)), so we have

$$\begin{aligned} \det \begin{bmatrix} b-a & c-a \\ (b-a)b & (c-a)c \end{bmatrix} &= (b-a)(c-a) \det \begin{bmatrix} 1 & 1 \\ b & c \end{bmatrix} \\ &= (b-a)(c-a)(c-b) \end{aligned}$$

which after rearranging and swapping signs we get the desired formula. □

(b) *Proof.*
We calculate following determinant by the same reasoning as in (a):

$$\begin{aligned} \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{n-1} & t_2^{n-1} & \dots & t_n^{n-1} \end{bmatrix} &= \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & t_2 - t_1 & \dots & t_n - t_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & t_2^{n-1} - t_1 t_2^{n-2} & \dots & t_n^{n-1} - t_1 t_n^{n-2} \end{bmatrix} \\ &= \det \begin{bmatrix} t_2 - t_1 & \dots & t_n - t_1 \\ \vdots & \ddots & \vdots \\ (t_2 - t_1)t_2^{n-2} & \dots & (t_n - t_1)t_n^{n-2} \end{bmatrix} \\ &= (t_2 - t_1) \dots (t_n - t_1) \det \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ t_2^{n-2} & \dots & t_n^{n-2} \end{bmatrix} \end{aligned}$$

Note that this matrix is just $(n-1)\times(n-1)$ equivalent of our starting matrix, so we can do this process again and get something like

$$(t_2-t_1)\dots(t_n-t_1)(t_3-t_2)\dots(t_n-t_2)\det\begin{bmatrix}1 & \dots & 1\\ \vdots & \ddots & \vdots\\ t_3^{n-3} & \dots & t_n^{n-3}\end{bmatrix}$$

Inductively, we get the final formula

$$\det\begin{bmatrix}1 & 1 & \dots & 1\\ t_1 & t_2 & \dots & t_n\\ \vdots & \vdots & \ddots & \vdots\\ t_1^{n-1} & t_2^{n-1} & \dots & t_n^{n-1}\end{bmatrix} = \prod_{i=1}^{n-1}\prod_{j=i+1}^{n}(t_j-t_i)$$

□

(c) *Proof.*
We want to find a polynomial $p(t)=a_nt^n+\cdots+a_1t+a_0$ such that $p(t_i)=s_i$ for some fixed pairs (t_i,s_i) for $0\le i\le n$. In other words, we have the system

$$S := \begin{bmatrix}s_0\\ \vdots\\ s_n\end{bmatrix} = \begin{bmatrix}a_0+a_1t_0+\cdots+a_nt_0^n\\ \vdots\\ a_0+a_1t_n+\cdots+a_nt_n^n\end{bmatrix} = \begin{bmatrix}1 & t_0 & \dots & t_0^n\\ \vdots & \vdots & \ddots & \vdots\\ 1 & t_n & \dots & t_n^n\end{bmatrix}\begin{bmatrix}a_0\\ \vdots\\ a_n\end{bmatrix} =: TA$$

Note T is the transpose of the matrix in (b), and by Exercise 4.5 we have $\det T=\det T^t=\prod_{j>i}(t_j-t_i)$, which is nonzero since the t_i's are necessarily distinct. Thus T is invertible and in particular we have a unique solution to our system, which are the coefficients for our unique polynomial. □

M.8

Consider a general system $AX = B$ of m linear equations in n unknowns, where m and n are not necessarily equal. The coefficient matrix A may have a left inverse L, a matrix such that $LA = I_n$. If so, we may try to solve the system as learn to do in school:

$$AX = B \implies LAX = LB \implies X = LB$$

But when we try to check our work by running the solution backward, we run into trouble: If $X = LB$, then $AX = ALB$. We seem to want L to be a right inverse, which isn't what was given.

(a) Work some examples to convince yourself that there is a problem here.

(b) Exactly what does the sequence of steps made above show? What would the existence of a right inverse show? Explain carefully.

Solution.

(a) Let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This has a left inverse, namely $L = \begin{bmatrix} 1 & 0 \end{bmatrix}$. However, if we let $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we get an issue:

$$X = LB = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \implies AX = 0 \neq B$$

Now if we let $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, then we have

$$X = LB = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \implies AX = B$$

so LB is in fact a solution to this particular system.

(b) The key behind these sequence of steps is that LB will be the **only** solution to the system **if one exists**. Indeed, if there exists $\hat{X}$ such that $A\hat{X} = B$, then the sequence exactly says $\hat{X} = LB$. However, we may be starting with a system with no solution, in which case writing $AX = B$ is not actually possible, hence the problem. [NB: In linear algebra terms, the injectivity of A as a linear operator is given by the existence of a left inverse, but we do not know a priori if B is in the image/column space of A]

Now supposing A has a right inverse R, i.e. $AR = I_m$, then the matrix $X = RB$ will always be a solution since

$$AX = A(RB) = (AR)B = B$$

However, we are not guaranteed that RB is the **only** solution to the system, and in fact there may be infinitely many solutions. [Again, in L.A. terms we have the surjectivity of A from the right inverse but the kernel/null space of A may vary]

M.9

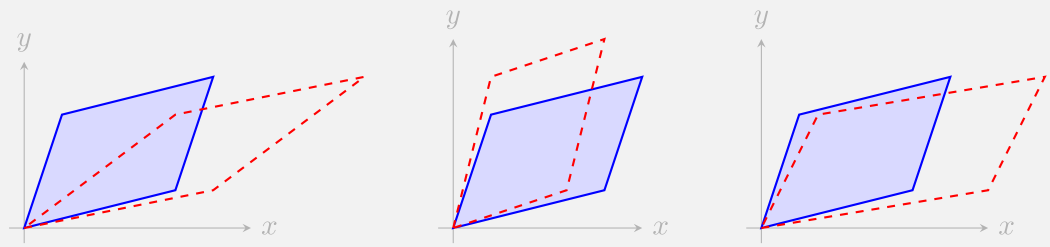
Let A be a real 2×2 matrix, and let A_1, A_2 be the columns of A. Let P be the parallelogram whose vertices are $0, A_1, A_2, A_1 + A_2$. Determine the effect of elementary row operations on the area of P, and use this to prove that the absolute value of the determinant of A is equal to the area of P.

Solution.
We go through each elementary row operation:

1. Row-addition: This corresponds to a shear mapping, which has no effect on area (this is most easily seen by starting with a rectangle, which after shearing will not change base or height, but rather just slant).

2. Row-swap: This corresponds to a reflection across the line $y = x$, which has no effect on area.

3. Row-scale: This corresponds to a scaling an entire axis, which will accordingly scale the area by that same factor.



shear: $R_1 \mapsto R_1 + R_2$ reflection: $R_1 \leftrightarrow R_2$ scale: $R_1 \mapsto 1.5R_1$

Above we have illustrated with the parallelogram $A = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$.
With this we claim that $\text{Area}(P) = |\det A|$.

Proof.
If A is not invertible, then $\det A = 0$ but this also means its columns are proportional. Thus the parallelogram degenerates to a single line or point, both of which have no area.

Now suppose A is invertible. Then we can write A as a product of elementary matrices, say $A = E_1 E_2 \dots E_n I$. We induct on n. If $n = 0$, then our starting parallelogram is the unit square, which has area $= 1 = |\det I|$. Now assume after n row operations we have $\text{Area}(P) = |\det E_1 \dots E_n|$ and consider one more row operation E. From the discussion above, we immediately get that the area of the changed parallelogram is accounted for by the factor $|\det E|$, which completes the induction. □

M.10

Let A, B be $m \times n$ and $n \times m$ matrices. Prove that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

Solution.

Proof.
$\Longrightarrow$: Suppose that $I_m - AB$ is invertible, say $(I_m - AB)^{-1} = C$. Then

$$\begin{aligned}(I_n + BCA)(I_n - BA) &= I_n - BA + BCA - BCABA \\ &= I_n - BA + B[C - CAB]A \\ &= I_n - BA + B[C(I_m - AB)]A \\ &= I_n - BA + B[I_m]A \\ &= I_n - BA + BA = I_n\end{aligned}$$

Hence $(I_n - BA)^{-1} = I_n + BCA$.

$\Longleftarrow$: Suppose that $I_n - BA$ is invertible, say $(I_n - BA)^{-1} = D$. Then

$$\begin{aligned}(I_m + ADB)(I_m - AB) &= I_m - AB + ADB - ADBAB \\ &= I_m - AB + A[D - DBA]B \\ &= I_m - AB + A[D(I_n - BA)]B \\ &= I_m - AB + A[I_n]B \\ &= I_m - AB + AB = I_m\end{aligned}$$

Hence $(I_m - AB)^{-1} = I_m + ADB$. □

M.11

A function $f(u,v)$ is *harmonic* if it satisfies the Laplace equation $\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = 0$. The Dirichlet problem asks for a harmonic function on a plane region R with prescribed values on the boundary. This exercise solves the discrete version of the Dirichlet problem.

Let f be a real-valued function whose domain of definition is the set of integers $\mathbb{Z}$. To avoid asymmetry, the discrete derivative is defined on the shifted integers $\mathbb{Z} + \frac{1}{2}$, as the first difference $f'(n + \frac{1}{2}) = f(n+1) - f(n)$. The discrete second derivative is back on the integers:

$$f''(n) = f'(n + \frac{1}{2}) - f'(n - \frac{1}{2}) = f(n+1) - 2f(n) + f(n-1)$$

Let $f(u,v)$ be a function whose domain is the lattice of points in the plane with integer coordinates. The formula for the discrete second derivative shows that the discrete version of the Laplace equation for f is

$$f(u+1,v) + f(u-1,v) + f(u,v+1) + f(u,v-1) - 4f(u,v) = 0$$

So f is harmonic if its value at a point (u,v) is the average of the values at its four neighbors.

A *discrete region* R in the plane is a finite set of integer lattice points. Its *boundary* ∂R is the set of lattice points that are not in R, but which are at a distance 1 from some point of R. We'll call R the *interior* of the region $\overline{R} = R \cup \partial R$. Suppose that a function β is given on the boundary ∂R. The discrete Dirichlet problem asks for a function f defined on $\overline{R}$ that is equal to β on the boundary and that satisfies the discrete Laplace equation at all points in the interior. This problem leads to a system of linear equations that we abbreviate as $LX = B$. To set the system up, we write β_{uv} for the given value of the function β at a boundary point. So $f(u,v) = \beta_{uv}$ at a boundary point (u,v). Let x_{uv} denote the unknown value of the function $f(u,v)$ at a point (u,v) of R. We order the points of R arbitrarily and assemble the unknowns x_{uv} into a column vector X. The coefficient matrix L expresses the discrete Laplace equation, except that when a point of R has some neighbors on the boundary, the corresponding terms will be the given boundary values. These terms are moved to the other side of the equation to form the vector B.

(a) When R is the set of five points $(0,0), (0,\pm 1), (\pm 1, 0)$, there are eight boundary points. Write down the system of linear equations in this case, and solve the Dirichlet problem when β is the function on ∂R defined by $\beta_{uv} = 0$ if $v \le 0$ and $\beta_{uv} = 1$ if $v > 0$.

(b) The *maximum principle* states that a harmonic function takes on its maximum value on the boundary. Prove the maximum principle for discrete harmonic functions.

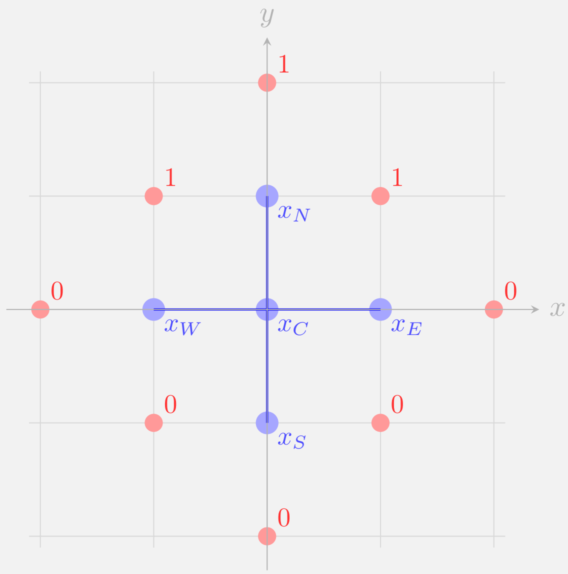
(c) Prove that the discrete Dirichlet problem has a unique solution for every region R and every boundary function β.

Solution.

(a) Per the directions above, we order the points of R arbitrarily as

$$x_C = f(0,0), \quad x_N = f(0,1), \quad x_S = f(0,-1), \quad x_E = f(1,0), \quad x_W = f(-1,0)$$

We can plot these points and the boundary values on the lattice below:



Hence we have Laplace equations:

$$\begin{cases} x_W + x_E + x_N + x_S - 4x_C & = 0 \\ 1 + 1 + 1 + x_C - 4x_N & = 0 \\ 0 + 0 + x_C + 0 - 4x_S & = 0 \\ x_C + 0 + 1 + 0 - 4x_E & = 0 \\ 0 + x_C + 1 + 0 - 4x_W & = 0 \end{cases} \implies \begin{cases} x_W + x_E + x_N + x_S - 4x_C & = 0 \\ x_C - 4x_N & = -3 \\ x_C - 4x_S & = 0 \\ x_C - 4x_E & = -1 \\ x_C - 4x_W & = -1 \end{cases}$$

which in matrix form $LX = B$ looks like

$$\begin{bmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -4 & 0 & 0 & 0 \\ 1 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & -4 & 0 \\ 1 & 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_C \\ x_N \\ x_S \\ x_E \\ x_W \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

We can solve each equation in terms of x_C:

$$x_N = \frac{x_C + 3}{4} \quad x_S = \frac{x_C}{4} \quad x_E = \frac{x_C + 1}{4} \quad x_W = \frac{x_C + 1}{4}$$

which when plugged back into the first equation gives

$$0 = \frac{(x_C + 1) + (x_C + 1) + (x_C + 3) + (x_C) - 16x_C}{4} = \frac{5 - 12x_C}{4} \implies x_C = \frac{5}{12}$$

and backsubstituting gives

$$x_N = \frac{41}{48} \quad x_S = \frac{5}{48} \quad x_E = \frac{17}{48} \quad x_W = \frac{17}{48}$$

(b) *Proof.*
Every value of a harmonic function in the interior of the region is the average of its neighbors, so the only way for the maximum value to occur in the region is if the function is constant. Otherwise values that are strictly less than the maximum will contribute to an average value that is also less than the maximum. In particular, this forces the maximum value to occur at the boundary. □

(c) *Proof.*
We want to show that $LX = B$ always has a unique solution. It suffices to show for the case $B = 0$, i.e. when all boundary values are zero, the only solution is the constant-zero function. Let f be a harmonic function on our region R. By (b), its maximum value is zero (since the entire boundary is zero), so $f(x) \leq 0$ for all $x \in R$.

For sake of contradiction, assume f is not constant-zero, i.e. there exists $\hat{x} \in R$ such that $f(\hat{x}) < 0$. Now note that if we let $g = -f$, then we have

$$g(u+1,v) + g(u-1,v) + g(u,v+1) + g(u,v-1) - 4g(u,v)$$

$$\begin{aligned} &= -[f(u+1,v) + f(u-1,v) + f(u,v+1) + f(u,v-1) - 4f(u,v)] \\ &= -[0] && (f \text{ harmonic}) \\ &= 0 \end{aligned}$$

Thus g is also harmonic, and for any boundary point $b \in \partial R$ we have $g(b) = -f(b) = -0 = 0$. Therefore g is also a solution to our system, but now note

$$g(\hat{x}) = -f(\hat{x}) > 0$$

thus g does not have a maximum value on the boundary, which is a contradiction. Thus $f(x) = 0$ for every $x \in R$ and f is the constant zero-function. □